CENTRAL UNIVERSITY OF HARYANA
End Semester Examinations January 2023

Programme	: Integrated B.Sc.-M.Sc. (Mathematics),	Session	: 2022-2023
Semester	: Third	Max. Time	: 3 Hours
Course Title	: Multivariable Calculus	Max. Marks	: 70
Course Code	: SBSMAT 03 03 01 C 5106		

Instructions:

1. Question no. 1 has seven sub parts and students need to answer any five. Each sub part carries two marks.
2. Question no. 2 to 6 have three sub parts and students need to answer any two sub parts of each question. Each sub part carries six marks.

1. (a) Find the equation for the tangent plane and normal line at the point P_0 on the given surface $x^2 + 2xy - y^2 + z^2 = 7$, $P_0(1, -1, 3)$.
   (b) Find the derivative of $f(x, y, z) = x^3 - xy^2 - z$ at $P_0(1, 1, 0)$ in the direction of $\vec{v} = 2\hat{i} - 3\hat{j} + 6\hat{k}$.
   (c) Evaluate $\nabla.(\vec{r} \times \vec{a})$, where $\vec{a}$ is constant vector and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.
   (d) Expand $e^x \sin y$ in powers of x and y as far as terms of the second degree.
   (e) If $u = x^y$, show that $\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial^3 u}{\partial x \partial y \partial x}$.
   (f) Show that $\int_C \phi \nabla \phi . d\vec{r} = 0$, C being a closed curve.
   (g) Change the order of integration in $I = \int_0^1 \int_{x^2}^{2-x} xy\, dy\, dx$ and hence evaluate the same.

2. (a) Express $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ as a function of u and v by using the Chain Rule, then evaluate them at the given point (u, v), where $z = 4e^x \ln y$, $x = \ln(u \cos v)$, $y = u \sin v$ and $(u, v) = (2, \frac{\pi}{4})$.
   (b) Let
   $$f(x,y) = \begin{cases} \frac{1}{4}(x^2 + y^2)\log(x^2 + y^2), & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}.$$
   Show that $f_{xy} = f_{yx}$ at all points (x, y). Also, show that f_{xy} and f_{yx} are not continuous.
   (c) The surfaces $f(x, y, z) = x^2 + y^2 - 2 = 0$ (A cylinder) and $g(x, y, z) = x + z - 4 = 0$ (A plane) meet in an ellipse E. Find parametric equation for the plane tangent to E at the point $P_0(1, 1, 3)$.

3. (a) State and prove Taylor's Theorem for functions of two variables.
   (b) Verify Euler's theorem for $\frac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$.
   (c) If $u = \frac{x}{y-z}$, $v = \frac{y}{z-x}$, $w = \frac{z}{x-y}$. Show that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$.

4. (a) Examine the function $x^3 + y^3 - 3axy$ for maxima and minima.
   (b) Find the maximum and minimum distance of the point $(3, 4, 12)$ from the sphere $x^2 + y^2 + z^2 = 1$ using Lagrange's method of multipliers.
   (c) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, find $\text{curl}\{(\vec{a} \times \vec{r})r^n\}$, where $r = |\vec{r}|$.

5. (a) Evaluate $\iint r \sin\theta\, dr d\theta$ over the area of the cardioid $r = a(1 + \cos(\theta))$ above the initial line.
   (b) Find the volume of the region S enclosed by the paraboloid $z = 6 - x^2 - y^2$ and $z = 5x^2 + 5y^2$.

(c) Using Spherical coordinate system, evaluate

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{\sqrt{4-x^2-y^2}} z^2\sqrt{x^2+y^2+z^2}\,dz\,dy\,dx.$$

6. (a) State and prove Gauss divergence theorem.

(b) Verify Stoke's theorem for $\vec{f} = (2x-y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}$ when S is the upper half of the surface of sphere $x^2+y^2+z^2=1$ and C is its boundary.

(c) State and prove Fundamental theorem for line integrals.